A Wronskian approach to the real τ-conjecture

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Plan

1. Introduction
   - Arithmetic circuits
   - τ-conjecture
2. real τ-conjecture
   - SPS
   - Conjecture
3. Results
   - Main tool
   - Upper bound
   - Application for different models

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Arithmetic circuits

Polynomial

$f(x, y) = x^2 - xy$

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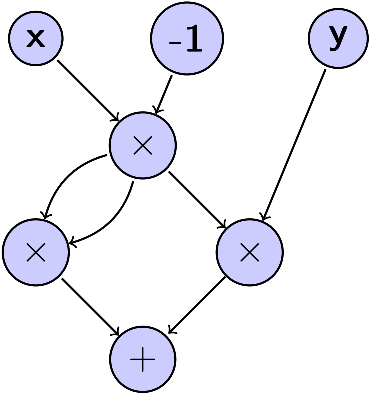

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Complexity of a polynomial

$\tau(P)$ = size of the smallest circuit computing P.

Example: $P(x) = x^2 - 1$: we get $\tau(P) = 4$

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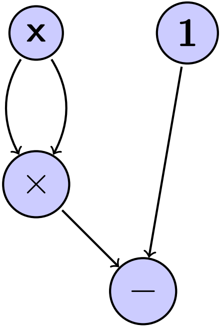

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- Example: $Det_n\left((x_{i,j})_{i,j \leq n}\right)$

Valiant's Conjecture

Class VP^0

(f_n): there exists c such that for all $n \geq 2$

- at most n^c variables
- $\tau(f_n) \leq n^c$
- formal degree is bounded by n^c

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$VP^0 \neq VNP^0$

τ-conjecture [Shub & Smale, 95]

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Remark:

- The conjecture is wrong for real roots:
  Chebyshev polynomials

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To bound the number of distinct real roots:

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  Bounded by $O(kt^m)$ by expanding

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Use of real analysis

The limited power of powering

And if the number of distinct f_{ij} is very small (constant?)?
Let us consider $f(X) = \sum_{i=1}^{k} \prod_{j=1}^{m} f_j^{\alpha_{ij}}(X)$ with f_j t-sparse polynomial.

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The Wronskian

Definition: Let $f_1, \ldots, f_k \in C^{k-1}(I)$ with $I \subset \mathbb{R}$. The *Wronskian* of the family is the determinant of the matrix:

$$W(f_1, \ldots, f_k) = \det \begin{bmatrix} f_1 & f_2 & \cdots & f_k \\ f_1' & f_2' & \cdots & f_k' \\ \vdots & \vdots & & \vdots \\ f_1^{(k-1)} & f_2^{(k-1)} & \ldots & f_k^{(k-1)} \end{bmatrix}$$

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First step

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With conditions over zeros of $W(f_1), W(f_1, f_2), \ldots, W(f_1, f_2, \ldots, f_k)$, find an upper bound of the number of zeros of $f_1 + \ldots + f_k$

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$$W\left(\left(\frac{f_2}{f_1}\right)', \dots, \left(\frac{f_q}{f_1}\right)'\right) = \left(\frac{1}{f_1}\right)^q W(f_1, \dots, f_q)$$

Results

Theorem (1, with P.Koiran and N.Portier)

Let $\Upsilon = \{x \in I | \exists s, W(f_1, \ldots, f_s)(x) = 0\}$.
Then

$$Z(a_1 f_1 + \ldots + a_k f_k) \leq (1 + |\Upsilon|)k - 1$$

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Theorem (2, with P.Koiran and N.Portier)

$$Z(f_1 + \ldots + f_k) \leq k - 1 + Z(W_k) + Z(W_{k-1}) + 2\sum_{j=1}^{k-2} Z(W_j)$$

Application: $\sum_{i=1}^{k}\prod_{j=1}^{m}(f_j)^{\alpha_{i,j}}$ with f_j sparse

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Theorem

If f is not zero, then it has at most $4ktm + 4\left(\mathrm{e}(1+t)\right)^{\frac{mk^2}{2}} = t^{O(mk^2)}$ distinct real roots.

Application: $\sum_{i=1}^{k}\prod_{j=1}^{m}(f_j)^{\alpha_{i,j}}$ with f_j of small degree

Theorem

$$f = \sum_{i=1}^{k}\prod_{j=1}^{m} f_j^{\alpha_{i,j}}$$

Then, $Z(f) \leq \frac{1}{3}k^3md + 2kmd + k = \frac{k^3md}{3}(1 + o(1))$

Some particular models

Avendaño's model

Corollary

Let $f = \sum_{i=1}^{k} c_i x^{\alpha_i}(ax+b)^{\beta_i}$*.*
Then $Z(f) = O(k^3)$*.*

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Li, Rojas and Wang's model

Corollary

Let $f = \sum_{i=1}^{k} a_i \prod_{j=1}^{m} (c_j x + d_j)^{\alpha_{i,j}}$.
Then $Z(f) = O(mk^3)$.

Open questions

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- Is the real τ-conjecture true?
- What about $fg + 1$? If f and g are t-sparse.

Thanks!