Knots, Braids and First Order Logic

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Outline

Knot

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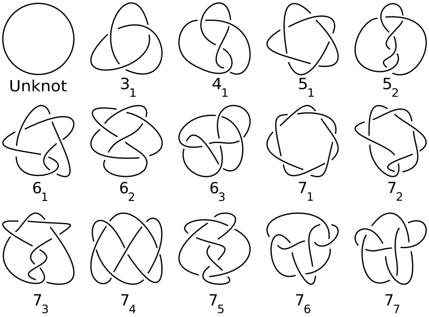

(Image source: Wikipedia)

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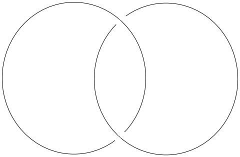

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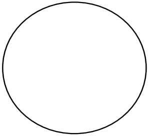

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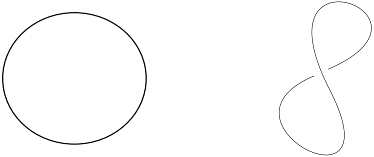

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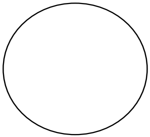

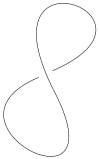

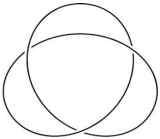

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- Ambient isotopy induces an equivalence relation between links.
- *Knot Equivalence Problem*: Given two knots K_1 and K_2, are they ambient isotopic to each other?
- *Unknotting Problem*: Given two knot, is it ambient isotopic to the unknot?

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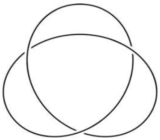

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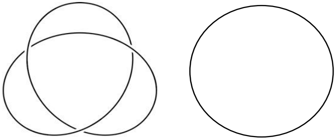

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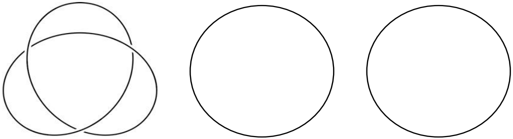

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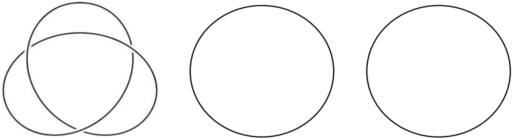

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Theorem

If K_1 and K_2 are knots (regarded as links), then $K_1 \equiv K_2$ if and only if K_1 is ambient isotopic to K_2.

Link Axioms (First Order Logic with Equality)

Consider a language with signature $(\cdot, T, \equiv, 1, \sigma, \bar{\sigma})$ such that $\cdot$ is a 2-function ,T is a 1-function, $\equiv$ is a 2-predicate, while 1,σ and $\bar{\sigma}$ are constants.

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- Group Axioms(for closed terms)
  1. $\forall x, y, z \quad (x \cdot (y \cdot z) = ((x \cdot y) \cdot z)$
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  3. $\forall x \quad x \cdot 1 = x$
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These axioms will be called *link axioms* and any model of these axioms will be called a *link model*.

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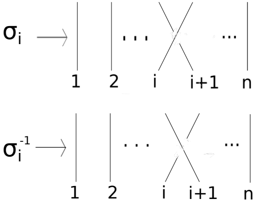

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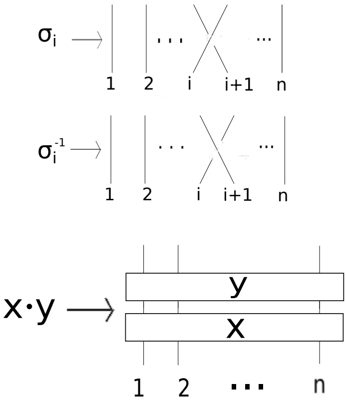

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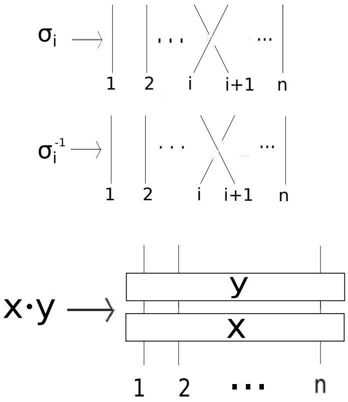

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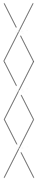

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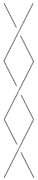

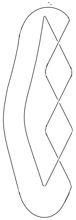

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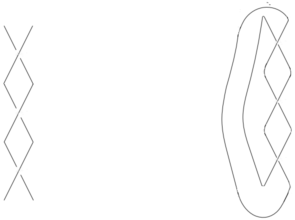

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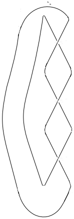

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Theorem (Alexander)

For every link L, there is an integer $m > 1$ and a braid $B \in B_m$ so that L is ambient isotopic to $\lambda(b, m)$.

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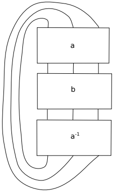

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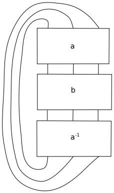

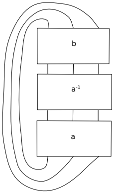

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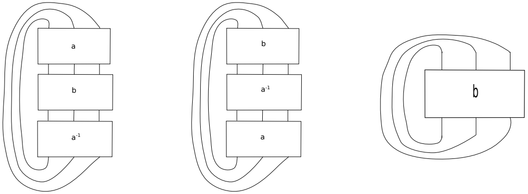

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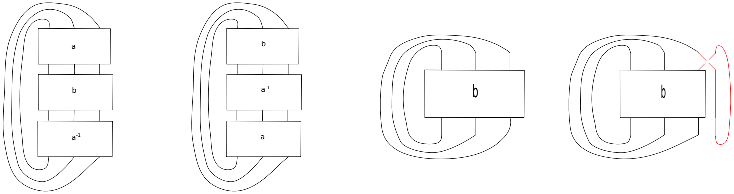

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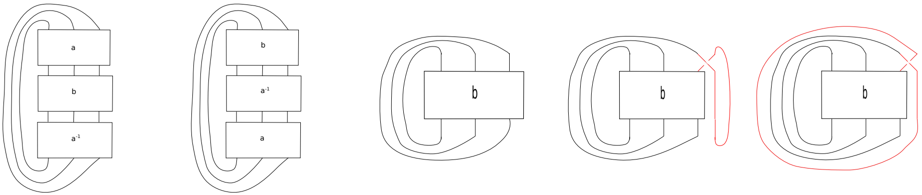

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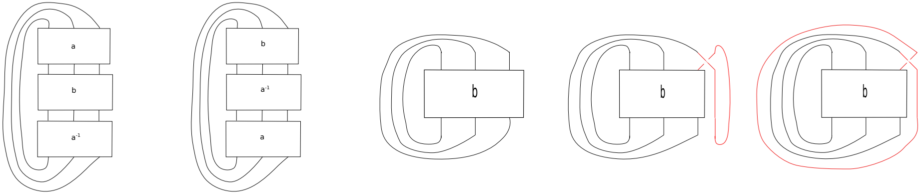

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Theorem (Markov)

For $i = 1, 2$, let $m_i > 1$ be integers and $b_i \in B_{m_i}$. Then the links $\lambda(b_1, m_1)$ and $\lambda(b_2, m_2)$ are isotopic if and only if $(b_1, m_1) \sim (b_2, m_2)$.

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Lemma

Two links are stably equivalent if and only if given $\lambda(b_1, m_1) = l_1$ and $\lambda(b_2, m_2) = l_2$, then $(b_1, m_1) \approx (b_2, m_2)$.

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Theorem (Main Theorem 1)

There is a surjective function $\Lambda : B_\infty \to \mathcal{L}$, where $\mathcal{L}$ is the set of links upto stable equivalence, such that for braids $b_1, b_2 \in B_\infty$, $\Lambda(b_1) = \Lambda(b_2)$ if and only if $b_1 \equiv b_2$.

- Group Axioms(for closed terms)
  1. $\forall x, y, z \quad (x \cdot (y \cdot z) = ((x \cdot y) \cdot z)$
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Theorem (Main Theorem 3)

$(B_\infty, T, \cdot, \equiv, \sigma_1, \sigma_1^{-1})$ is a canonical model for link axioms.

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- However in the finite models, all the closed terms are markov equivalent to each other.
- This formulation enables us to formulate knot theory in terms of first order logic and thus renders it implementable in Automated Theorem Provers.

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