An elementary recursive bound for the Positivstellensatz

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Positivstellensatz (Krivine '64, Stengle '74)

- **K** an ordered field, **R** a real closed extension of **K**,
- $P_1, \ldots, P_s \in \mathbf{K}[x_1, \ldots, x_k]$,
- $I_{\neq}, I_{\geq}, I_{=} \subset \{1, \ldots, s\}$,

$$\mathcal{H}(x): \begin{cases} P_i(x) \neq 0 & \text{for} \quad i \in I_{\neq} \\ P_i(x) \geq 0 & \text{for} \quad i \in I_{\geq} \\ P_i(x) = 0 & \text{for} \quad i \in I_{=} \end{cases} \quad \text{no solution in } \mathbf{R}^k \quad \Longleftrightarrow$$

$$\exists \quad S = \prod_{i \in I_{\neq}} P_i^{2e_i}, \qquad N = \sum_{I \subset I_{\geq}} \Big(\sum_j k_{I,j} Q_{I,j}^2 \Big) \prod_{i \in I} P_i \quad (k_{I,j} > 0),$$

$$Z \in \langle P_i \mid i \in I_{=} \rangle \subset \mathbf{K}[x]$$

such that $\underbrace{S}_{>0} + \underbrace{N}_{\geq 0} + \underbrace{Z}_{=0} = 0.$ $\qquad \leftarrow \quad \downarrow \mathcal{H} \downarrow$

Example:

$$\begin{cases} x & \neq & 0 \\ y - x^2 - 1 & \geq & 0 \\ xy & = & 0 \end{cases} \quad \text{no solution in } \mathbf{R}^2$$

$\downarrow \; x \neq 0, \;\; y - x^2 - 1 \geq 0, \;\; xy = 0 \; \downarrow$:

$$\underbrace{x^2}_{>0} + \underbrace{x^2(y - x^2 - 1) + x^4}_{\geq 0} + \underbrace{(-x^2 y)}_{=0} = 0.$$

Considered systems:

$$\begin{cases} P_i(x) & \neq & 0 & \text{for} & i \in I_{\neq} \\ P_i(x) & \geq & 0 & \text{for} & i \in I_{\geq} \\ P_i(x) & = & 0 & \text{for} & i \in I_{=} \end{cases}$$

For any given system:

$$\begin{aligned} P \leq 0 &\longrightarrow -P \geq 0, \\ P > 0 &\longrightarrow P \neq 0, \quad P \geq 0 \\ P < 0 &\longrightarrow -P \neq 0, \quad -P \geq 0 \end{aligned}$$

Algebraically closed case: Nullstellensatz

- $\mathbf{K}$ a field, $\mathbf{C}$ an algebraically closed extension of $\mathbf{K}$,
- $P_1, \ldots, P_s \in \mathbf{K}[x_1, \ldots, x_k]$,
- $I_{\neq}, I_{=} \subset \{1, \ldots, s\}$,

$$\begin{cases} P_i(x) \neq 0 & \text{for} \quad i \in I_{\neq} \\ P_i(x) = 0 & \text{for} \quad i \in I_{=} \end{cases} \quad \text{no solution in } \mathbf{C}^k \quad \Longleftrightarrow$$

$$\exists \quad S = \prod_{i \in I_{\neq}} P_i^{e_i}, \qquad Z \in \langle P_i \mid i \in I_{=} \rangle \subset \mathbf{K}[x] \qquad \text{such that}$$

$$\underbrace{S}_{\neq 0} + \underbrace{Z}_{= 0} = 0.$$

Algebraically closed case: Nullstellensatz

$$S \quad + \quad Z \quad = \quad 0.$$

- Optimal bound (Jelonek '05):

$$S = \prod_{i \in I_{\neq}} P_i^{e_i} \qquad \text{with} \qquad e_i \leq d^k \qquad (\deg P_i \leq d).$$

$$Z = \sum_{i \in I_{=}} P_i Q_i \qquad \text{with} \qquad \deg P_i Q_i \leq (1 + d|I_{\neq}|)d^k.$$

- from the degree bound, $Z = \sum_{i \in I_{=}} P_i Q_i$ can be obtained solving a linear system.

Real closed case: Positivstellensatz

$$S \quad + \quad N \quad + \quad Z \quad = \quad 0.$$

- Constructive proof (Lombardi '90) based in Hörmander algorithm:

  degree bound $\rightarrow$ primitive recursive in k, s and d.

- Our work (in progress): New constructive proof based in cilindrical decomposition:

  degree bound $\rightarrow$ elementary recursive in k, s and d.

  (exponential tower of height 5)

Hilbert 17th problem

$$P \geq 0 \text{ in } \mathbb{R}^k \implies \exists P = \sum_i G_i^2, \quad G_i \in \mathbb{R}(x_1, \ldots, x_k) \, ?$$

- Artin '27: Affirmative answer.
- Positivstellensatz implies Hilbert 17th problem:

$$\left\{ P(x) < 0 \quad \text{no solution} \right. \iff \left\{ \begin{array}{rcl} -P(x) & \neq & 0 \\ -P(x) & \geq & 0 \end{array} \right. \quad \text{no solution}$$

$$\iff \quad P^{2e} + \sum_i Q_i^2 - (\sum_j R_j^2) P = 0 \quad \iff$$

$$P = \frac{P^{2e} + \sum_i Q_i^2}{\sum_j R_j^2} = \frac{(P^{2e} + \sum_i Q_i^2)(\sum_j R_j^2)}{(\sum_j R_j^2)^2}.$$

Incompatibilities

$$\mathcal{H}(x): \begin{cases} P_i(x) & \neq & 0 & \text{for} & i \in I_{\neq} \\ P_i(x) & \geq & 0 & \text{for} & i \in I_{\geq} \\ P_i(x) & = & 0 & \text{for} & i \in I_{=} \end{cases}$$

$$\downarrow \mathcal{H} \downarrow : \quad \underbrace{S}_{>0} + \underbrace{N}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

with

$S \in \left\{ \prod_{i \in I_{\neq}} P_i^{2e_i} \right\}$ $\leftarrow$ monoid associated to $\mathcal{H}$

$N \in \left\{ \sum_{I \subset I_{\geq}} \left(\sum_j k_{I,j} Q_{I,j}^2 \right) \prod_{i \in I} P_i \right\}$ $\leftarrow$ cone associated to $\mathcal{H}$

$Z \in \langle P_i \mid i \in I_{=} \rangle$ $\leftarrow$ ideal associated to $\mathcal{H}$

	monoid part		cone part		ideal part		
$\downarrow \mathcal{H} \downarrow :$	$\underbrace{S}_{>0}$	$+$	$\underbrace{N}_{\geq 0}$	$+$	$\underbrace{Z}_{=0}$	$=$	0
$\downarrow \mathcal{H},\ A \neq 0 \downarrow :$	$\underbrace{A^{2e}S}_{>0}$	$+$	$\underbrace{N}_{\geq 0}$	$+$	$\underbrace{Z}_{=0}$	$=$	0
$\downarrow \mathcal{H},\ A \geq 0 \downarrow :$	$\underbrace{S}_{>0}$	$+$	$\underbrace{N + N'A}_{\geq 0}$	$+$	$\underbrace{Z}_{=0}$	$=$	0
$\downarrow \mathcal{H},\ A = 0 \downarrow :$	$\underbrace{S}_{>0}$	$+$	$\underbrace{N}_{\geq 0}$	$+$	$\underbrace{Z + WA}_{=0}$	$=$	0

Weak inferences

$$A > 0, \quad B \geq 0 \quad \Longrightarrow \quad A + B > 0.$$

Let $\mathcal{H}$ be a system of sign conditions.

$$\downarrow \mathcal{H}, \ A + B > 0 \downarrow \quad \longrightarrow \quad \left\{ \begin{array}{lcl} \mathcal{H}(x) & & \\ A(x) + B(x) & > & 0 \end{array} \right. \quad \text{no solution}$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\downarrow \mathcal{H}, \ A > 0, \ B \geq 0 \downarrow \quad \longleftarrow \quad \left\{ \begin{array}{lcl} \mathcal{H}(x) & & \\ A(x) & > & 0 \\ B(x) & \geq & 0 \end{array} \right. \quad \text{no solution}$$

$$A > 0, \quad B \geq 0 \quad \vdash \quad A + B > 0$$

$$A > 0, \quad B \geq 0 \quad \vdash \quad A + B > 0$$

$$\downarrow \mathcal{H}, \ A + B > 0 \downarrow \quad \rightarrow \quad \downarrow \mathcal{H}, \ A + B \neq 0, \ A + B \geq 0 \downarrow$$

$$\underbrace{(A+B)^{2e}S}_{>0} + \underbrace{N + N'(A+B)}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

$$\downarrow$$

$$\underbrace{A^{2e}S}_{>0} + \underbrace{\sum_{i=0}^{2e-1} \binom{2e}{i} A^i B^{2e-1} S + N + N'A + N'B}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

$$\downarrow \mathcal{H}, \ A \neq 0, \ A \geq 0, \ B \geq 0 \downarrow \quad \rightarrow \quad \downarrow \mathcal{H}, \ A > 0, \ B \geq 0 \downarrow$$

$$A > 0, \quad AB \geq 0 \quad \vdash \quad B \geq 0$$

$$\downarrow \mathcal{H}, \quad B \geq 0 \downarrow$$

$$\underbrace{S}_{>0} + \underbrace{N + N'B}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

$$\downarrow$$

$$\underbrace{A^2 S}_{>0} + \underbrace{NA^2 + N'A(AB)}_{\geq 0} + \underbrace{ZA^2}_{=0} = 0$$

$$\downarrow \mathcal{H}, \quad A \neq 0, \quad A \geq 0, \quad AB \geq 0 \downarrow \quad \rightarrow \quad \downarrow \mathcal{H}, \quad A > 0, \quad AB \geq 0 \downarrow$$

$$A \neq 0 \quad \vdash \quad A < 0 \quad \vee \quad A > 0$$

Let $\mathcal{H}$ be a system of sign conditions.

$$\downarrow \mathcal{H}, \ A < 0 \downarrow \longrightarrow \left\{ \begin{array}{l} \mathcal{H}(x) \\ A(x) \end{array} \right. \begin{array}{c} \\ < \end{array} \begin{array}{l} \\ 0 \end{array} \quad \text{no solution}$$

$$\downarrow \mathcal{H}, \ A > 0 \downarrow \longrightarrow \left\{ \begin{array}{l} \mathcal{H}(x) \\ A(x) \end{array} \right. \begin{array}{c} \\ > \end{array} \begin{array}{l} \\ 0 \end{array} \quad \text{no solution}$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \downarrow \mathcal{H}, \ A \neq 0 \downarrow & \longleftarrow & \left\{ \begin{array}{l} \mathcal{H}(x) \\ A(x) \end{array} \right. \begin{array}{c} \\ \neq \end{array} \begin{array}{l} \\ 0 \end{array} \quad \text{no solution} \end{array}$$

$$A \neq 0 \quad \vdash \quad A < 0 \quad \vee \quad A > 0$$

$$\downarrow \mathcal{H},\ A < 0 \downarrow \qquad\qquad \downarrow \mathcal{H},\ A > 0 \downarrow$$

$$\underbrace{A^{2e_1} S_1}_{>0} + \underbrace{N_1 - N_1' A}_{\geq 0} + \underbrace{Z_1}_{=0} = 0 \qquad \underbrace{A^{2e_2} S_2}_{>0} + \underbrace{N_2 + N_2' A}_{\geq 0} + \underbrace{Z_2}_{=0} = 0$$

$$A^{2e_1} S_1 + N_1 + Z_1 = N_1' A \qquad A^{2e_2} S_2 + N_2 + Z_2 = -N_2' A$$

$$\searrow \qquad\qquad \swarrow$$

$$A^{2e_1+2e_2} S_1 S_2 + N_3 + Z_3 = -N_1' N_2' A^2$$

$$\underbrace{A^{2e_1+2e_2} S_1 S_2}_{>0} + \underbrace{N_1' N_2' A^2 + N_3}_{\geq 0} + \underbrace{Z_3}_{=0} = 0$$

$$\downarrow \mathcal{H},\ A \neq 0 \downarrow$$

$$\frac{\begin{array}{ccc} & A = 0 \quad \vee \quad A \neq 0 \\ A \neq 0 \quad \Longrightarrow & A < 0 \quad \vee \quad A > 0 \end{array}}{A = 0 \quad \vee \quad A < 0 \quad \vee \quad A > 0}$$

Similarly,

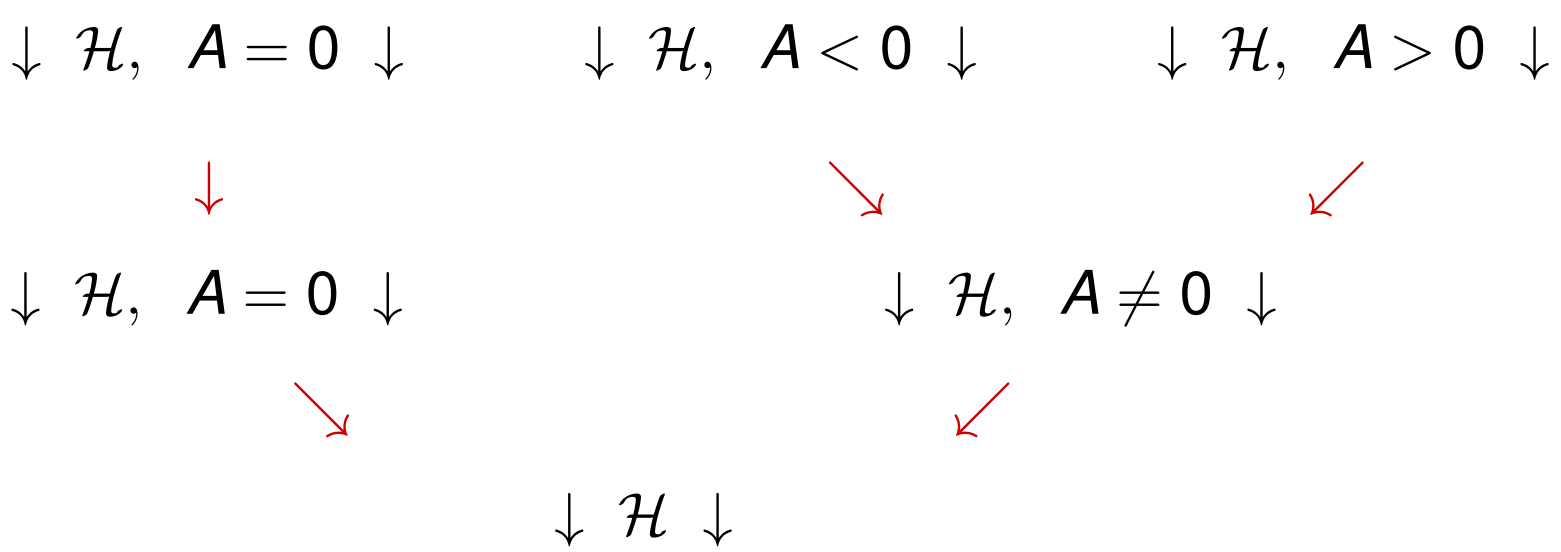
$$\frac{\begin{array}{ccc} \vdash & A = 0 \quad \vee \quad A \neq 0 \\ A \neq 0 \quad \vdash & A < 0 \quad \vee \quad A > 0 \end{array}}{\vdash \quad A = 0 \quad \vee \quad A < 0 \quad \vee \quad A > 0}$$

$$\vdash \quad A = 0 \quad \vee \quad A \neq 0$$

$$A \neq 0 \quad \vdash \quad A < 0 \quad \vee \quad A > 0$$

$$\vdash \quad A = 0 \quad \vee \quad A < 0 \quad \vee \quad A > 0$$

as follows:



Weak Existence

$$\exists t \quad | \quad t\,A = 1 \quad \vdash \quad A \neq 0$$

$$\downarrow \mathcal{H},\ A \neq 0 \downarrow$$

$$\underbrace{A^{2e}S}_{>0} + \underbrace{N}_{\geq 0} + \underbrace{Z}_{=0} = 0$$

$$\downarrow$$

$$\underbrace{S}_{>0} + \underbrace{t^{2e}N}_{\geq 0} + \underbrace{t^{2e}Z + ((tA)^{2e} - 1)S}_{=0} = 0$$

$\downarrow \mathcal{H},\ tA = 1 \downarrow$ $\leftarrow$ involves the new variable t

$$A \neq 0 \quad \vdash \quad \exists t \quad | \quad t\,A = 1$$

$$\downarrow \mathcal{H},\ \ tA = 1 \downarrow \quad \leftarrow \quad \text{involves variable } t$$

$$\underbrace{S}_{>0} + \underbrace{\sum \left(\sum k_{l,j} Q_{l,j}^2(t) \right) \prod P_i}_{\geq 0} + \underbrace{\sum W_j(t) P_j \ + \ W(t)(tA - 1)}_{=0} = 0$$

$$\downarrow$$

$$\underbrace{A^{2e} S}_{>0} \ + \ \underbrace{N}_{\geq 0} \ + \ \underbrace{Z}_{=0} \ = \ 0$$

$$\downarrow \mathcal{H},\ \ A \neq 0 \downarrow \quad \leftarrow \quad \text{variable } t \text{ is eliminated}$$

Obtaining the incompatibility: example

$$P_0(x_1, x_2) = x_1^2 + x_2^2, \qquad P_1(x_1, x_2) = x_1^2 + x_2^2 - 1.$$

$$\mathcal{H} = \begin{cases} P_0(x_1, x_2) & = & 0 \\ P_1(x_1, x_2) & \geq & 0 \end{cases} \quad \text{no solution in } \mathbf{R}^2.$$

Realizable systems of strict sign conditions for P_0, P_1:

$$\begin{cases} P_0(x_1, x_2) & > & 0 \\ P_1(x_1, x_2) & > & 0 \end{cases} \qquad \begin{cases} P_0(x_1, x_2) & > & 0 \\ P_1(x_1, x_2) & = & 0 \end{cases}$$

$$\begin{cases} P_0(x_1, x_2) & > & 0 \\ P_1(x_1, x_2) & < & 0 \end{cases} \qquad \begin{cases} P_0(x_1, x_2) & = & 0 \\ P_1(x_1, x_2) & < & 0 \end{cases}$$

$$P_0(x_1, x_2) = x_1^2 + x_2^2, \qquad P_1(x_1, x_2) = x_1^2 + x_2^2 - 1.$$

$$\mathcal{H} = \begin{cases} P_0(x_1, x_2) & = & 0 \\ P_1(x_1, x_2) & \geq & 0 \end{cases}$$

$$\vdash \quad \begin{array}{l} (P_0 > 0,\ P_1 > 0) \quad \vee \quad (P_0 > 0,\ P_1 = 0) \quad \vee \\ (P_0 > 0,\ P_1 < 0) \quad \vee \quad (P_0 = 0,\ P_1 < 0). \end{array}$$

- $\downarrow \mathcal{H},\ P_0 > 0,\ P_1 > 0 \downarrow :$ $\qquad P_0^2 + 0 + (-P_0^2) = 0.$
- $\downarrow \mathcal{H},\ P_0 > 0,\ P_1 = 0 \downarrow :$ $\qquad P_0^2 + 0 + (-P_0^2) = 0.$
- $\downarrow \mathcal{H},\ P_0 > 0,\ P_1 < 0 \downarrow :$ $\qquad P_0^2 + 0 + (-P_0^2) = 0.$
- $\downarrow \mathcal{H},\ P_0 = 0,\ P_1 < 0 \downarrow :$ $\qquad (-P_1)^2 + P_1(-P_1) + 0 = 0.$

$$P_0(x_1, x_2) = x_1^2 + x_2^2, \qquad P_1(x_1, x_2) = x_1^2 + x_2^2 - 1.$$

$$\begin{aligned} & x_1 < -1 \quad \vee \quad x_1 = -1 \quad \vee \quad -1 < x_1 < 0 \quad \vee \\ \vdash \quad & x_1 = 0 \quad \vee \quad 0 < x_1 < 1 \quad \vee \quad x_1 = 1 \quad \vee \\ & 1 < x_1. \end{aligned}$$

$$x_1 < -1 \quad \vdash \quad P_0 > 0, \quad P_1 > 0.$$

$$x_1 = -1 \quad \vdash \quad (P_0 > 0, \quad P_1 > 0) \quad \vee \quad (P_0 > 0, \quad P_1 = 0).$$

$$\begin{aligned} -1 < x_1 < 0 \quad \vdash \quad & (P_0 > 0, \quad P_1 > 0) \quad \vee \quad (P_0 > 0, \quad P_1 = 0) \quad \vee \\ & (P_0 > 0, \quad P_1 < 0). \end{aligned}$$

$$\vdots$$

$$P_0(x_1, x_2) = x_1^2 + x_2^2, \qquad P_1(x_1, x_2) = x_1^2 + x_2^2 - 1.$$

$$-1 < x_1 < 0 \quad \vdash \quad \exists(a, b, t_1, t_2) \quad \left| \begin{array}{ll} -2ax_1 = 0, & a^2 + b^2 = x_1^2, \\ -(t_1 + t_2) = 0, & t_1 t_2 = x_1^2 - 1, \\ b \neq 0, & t_1 < t_2. \end{array} \right.$$

- $P_0(x_1, x_2) = (x_2 - a)^2 + b^2$ with $b \neq 0$,
- $P_1(x_1, x_2) = (x_2 - t_1)(x_2 - t_2)$ with $t_1 < t_2$.

Obtaining the incompatibility: general case

- $\mathcal{P} = P_1, \ldots, P_s \subset \mathbf{R}[x_1, \ldots, x_k]$,
- $I_{\neq}, I_{\geq}, I_{=} \subset \{1, \ldots, s\}$,

$$\mathcal{H} = \begin{cases} P_i(x) & \neq & 0 & \text{for} & i \in I_{\neq} \\ P_i(x) & \geq & 0 & \text{for} & i \in I_{\geq} \\ P_i(x) & = & 0 & \text{for} & i \in I_{=} \end{cases}$$

no solution in $\mathbf{R}^k$.

- For any realizable strict sign condition σ on $\mathcal{P}$,

$$\downarrow \mathcal{H}, \ \ \text{sign}(\mathcal{P}) = \sigma \ \downarrow$$

- a weak inference

$$\vdash \bigvee_{\sigma \text{ realizable for } \mathcal{P}} \text{sign}(\mathcal{P}) = \sigma.$$

- For any realizable strict sign condition σ on $\mathcal{P}$,

$$\downarrow \mathcal{H}, \ \ \text{sign}(\mathcal{P}) = \sigma \downarrow$$

$\mathcal{H}$ is given by a non relizable sign condition on $\mathcal{P}$. We have at least one conflict of signs:

$\mathcal{H}$	$\text{sign}(\mathcal{P}) = \sigma$
$P_i > 0$	$P_i = 0$
$P_i > 0$	$P_i < 0$
$P_i \geq 0$	$P_i < 0$
$P_i = 0$	$P_i > 0$
$P_i = 0$	$P_i < 0$

$\mathcal{H}$	$\text{sign}(\mathcal{P}) = \sigma$
$P_i \leq 0$	$P_i > 0$
$P_i < 0$	$P_i > 0$
$P_i < 0$	$P_i = 0$
$P_i \neq 0$	$P_i = 0$

$$P_i^2 - P_i^2 = 0$$

- $\vdash \bigvee_{\sigma \text{ realizable for } \mathcal{P}} \operatorname{sign}(\mathcal{P}) = \sigma.$

$$\begin{aligned} \mathcal{P} = \mathcal{P}_k &\subset \mathbf{R}[X_1, \ldots, X_k] \\ \mathcal{P}_{k-1} &\subset \mathbf{R}[X_1, \ldots, X_{k-1}] \\ &\vdots \\ \mathcal{P}_1 &\subset \mathbf{R}[X_1] \\ \mathcal{P}_0 &= \emptyset \end{aligned}$$

For every realizable strict sign condition τ for $\mathcal{P}_{j-1}$,

$$\operatorname{sign}(\mathcal{P}_{j-1}) = \tau \vdash \bigvee_{\substack{\sigma \text{ realizable for } \mathcal{P}_j \\ \text{with } \operatorname{sign}_{(\mathcal{P}_{j-1})} = \tau}} \operatorname{sign}(\mathcal{P}_j) = \sigma.$$

1) $\text{sign}(\mathcal{P}_{j-1}) = \tau \quad \vdash$ Factorization of family $\mathcal{P}_j$ and for all $Q_1, Q_2 \in \mathcal{P}_j$, sign of every derivative of Q_2 at real roots of Q_1.

2) Factorization of family $\mathcal{P}_j$ and for all $Q_1, Q_2 \in \mathcal{P}_j$, sign of every derivative of Q_2 at real roots of Q_1. $\vdash$ Factorization of family $\mathcal{P}_j$ and order of real roots.

3) Factorization of family $\mathcal{P}_j$ and order of real roots. $\vdash \bigvee_{\sigma \text{ realizable for } \mathcal{P}_j \text{ with } \text{sign}(\mathcal{P}_{j-1})=\tau} \text{sign}(\mathcal{P}_j) = \sigma.$

1) $\operatorname{sign}(\mathcal{P}_{j-1}) = \tau \quad \vdash$ Factorization of family $\mathcal{P}_j$ and for all $Q_1, Q_2 \in \mathcal{P}_j$, sign of every derivative of Q_2 at real roots of Q_1.

Hermite Theory: $q = \deg_{x_j} Q_1$, $\operatorname{Her}(Q_1, Q_2) \in \mathbf{K}[x_1, \ldots, x_{j-1}]^{q \times q}$

$$\operatorname{Her}(Q_1, Q_2)_{\ell_1,\ell_2} = \text{Trace of mult. by } x_j^{\ell_1+\ell_2} Q_2 \text{ in } \mathbf{K}[x]/\langle Q_1 \rangle.$$

$$\text{Signature}(\operatorname{Her}(Q_1, Q_2)_{\ell_1,\ell_2}) =$$
$$= \#\{t \in \mathbf{R} \mid Q_1(t) = 0,\ Q_2(t) > 0\} -$$
$$-\#\{t \in \mathbf{R} \mid Q_1(t) = 0,\ Q_2(t) < 0\}.$$

2) Factorization of family $\mathcal{P}_j$ and for all $Q_1, Q_2 \in \mathcal{P}_j$, sign of every derivative of Q_2 at real roots of Q_1. $\vdash$ Factorization of family $\mathcal{P}_j$ and order of real roots.

Thom encoding of real algebraic numbers:

Example: P monic, $\deg P = 4$.

$$\begin{array}{llll} P(t_1) < 0, & P(t_2) = 0, \\ P'(t_1) = 0, & P'(t_2) > 0, \\ P''(t_1) > 0, & P''(t_2) > 0, \\ P'''(t_1) < 0, & P'''(t_2) < 0, \end{array} \implies t_1 < t_2$$

3) Factorization of family $\mathcal{P}_j$ and order of real roots. $\vdash$ $\bigvee_{\sigma \text{ realizable for } \mathcal{P}_j \text{ with } \operatorname{sign}(\mathcal{P}_{j-1})=\tau} \operatorname{sign}(\mathcal{P}_j) = \sigma.$

Partition of the real line according to real roots.

Thanks!