Moment matrices for root finding

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Algebraic method for solving polynomial equations

- $\mathbb{K}[x_1, \ldots, x_n] = \mathbb{K}[\mathbf{x}] = R$ the multivariate polynomial ring over the field $\mathbb{K}$.
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Normal form algorithms: **Gröbner basis**, **H-basis**, **Janet basis**, **Border basis** ...

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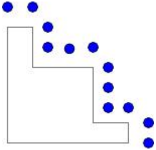

- B a set of monomials **connected** to 1 ($1 \in B$, $\forall m \in B \setminus \{1\}$ $\exists m' \in B$, $i \in [1, n]$ st. $m = m' x_i$).
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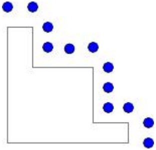

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Theorem

Let $d \geq 2$, let B be a subset of $\mathcal{M}$ connected to 1, let $\pi : \langle B^+ \rangle_{\leq d} \to \langle B \rangle_{\leq d}$ be a projection and let F be the rewriting family of π.

The following conditions are equivalent:

1. $(M_i \circ M_j - M_j \circ M_i)_{|\langle B \rangle_{\leq d-2}} = 0$ *for* $1 \leq i < j \leq n$,
2. *there exists a unique projection* $\tilde{\pi} : R_{\leq d} \to \langle B \rangle_{\leq d}$ *such that the restriction of* $\tilde{\pi}$ *to* $\langle B^+ \rangle_{\leq d}$ *is* π *and* $\ker \tilde{\pi} = \langle F^{\langle \leq d \rangle} \rangle$,

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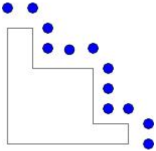

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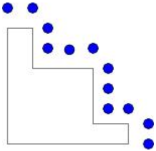

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Hypothesis: $\mathcal{V}_{\overline{\mathbb{K}}}(I) = \{\zeta_1, \ldots, \zeta_r\} \Leftrightarrow \mathcal{A} = \mathbb{K}[\mathbf{x}]/I$ of dimension $D < \infty$ over $\mathbb{K}$.

Operators of multiplication:

$$\begin{array}{rcl} \mathcal{M}_a : \mathcal{A} & \to & \mathcal{A} \\ u & \mapsto & a\, u \end{array} \qquad \begin{array}{rcl} \mathcal{M}_a^{\mathsf{t}} : \widehat{\mathcal{A}} & \to & \widehat{\mathcal{A}} \\ \Lambda & \mapsto & a \cdot \Lambda = \Lambda \circ M_a \end{array}$$

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In practice, take some a in $\langle x_1, \ldots, x_n \rangle$.

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$$\Lambda = \sum_{\alpha \in \mathbb{N}^n} \Lambda(\mathbf{x}^\alpha) \mathbf{d}^\alpha$$

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Example: $x_1 \cdot \mathbf{d}_1^{\alpha_1}\mathbf{d}_2^{\alpha_2}\cdots\mathbf{d}_n^{\alpha_n} = \mathbf{d}_1^{\alpha_1-1}\mathbf{d}_2^{\alpha_2}\cdots\mathbf{d}_n^{\alpha_n}$ if $\alpha_1 > 0$ and 0 otherwise.

Examples

- $p \mapsto p(\zeta)$ represented by the series $\mathbf{1}_\zeta = \sum_{\alpha\in\mathbb{N}^n} \zeta^\alpha\, \mathbf{d}^\alpha$.
- $p \mapsto \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}(p)(0)$ represented by $\alpha!\, \mathbf{d}^\alpha$.
- $p \mapsto$ coefficient of x^α in $\pi(p)$.
- $p \mapsto \int_\Omega p\, d\mu$.

Our objective

Exploit the properties of the dual

$$\mathcal{A}^* = \{\Lambda : \mathbb{K}[\mathbf{x}] \to \mathbb{K} \mid \Lambda(I) = 0\} = I^{\perp}$$

of $\mathcal{A} = \mathbb{K}[\mathbf{x}]/I$ to find the roots $\mathcal{V}(I) = \{\zeta_1, \ldots, \zeta_r\}$.

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Outline

1. Properties
2. Applications

Moment matrices and Hankel operators

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$$\begin{array}{rcl} H_\Lambda^{E_1,E_2} : E_1 & \rightarrow & E_2^* \\ p & \mapsto & p \cdot \Lambda \end{array}$$

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- If $E = \mathbb{K}[x_1, \ldots, x_n]$, we define the **Hankel operator**:

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$$\Lambda : p \mapsto \sum_{i=1}^{r'} \mathbf{1}_{\zeta_i} \cdot \theta_i(\partial_{x_1}, \ldots, \partial_{x_n})(p)$$

for some $\zeta_i \in \mathbb{C}^n$ and some differential polynomials θ_i with

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- If Λ is supported on points, then $\mathcal{A}_\Lambda = R/I_\Lambda$ is a Gorenstein algebra:
  1. $\mathcal{A}_\Lambda^* = \mathcal{A}_\Lambda \cdot \Lambda$ (free module of rank 1).
  2. $(a, b) \mapsto \Lambda(ab)$ is non-degenerate in $\mathcal{A}_\Lambda$.
  3. $\mathrm{Hom}_{\mathcal{A}_\Lambda}(\mathcal{A}_\Lambda^*, \mathcal{A}_\Lambda) = \mathcal{D} \cdot \mathcal{A}_\Lambda$ where $\mathcal{D} = \sum_{i=1}^{r} b_i \otimes \omega_i$ for $(b_i)_{1 \leqslant i \leqslant r}$ a basis of $\mathcal{A}_\Lambda$ and $(\omega_i)_{1 \leqslant i \leqslant r}$ its dual basis for Λ.

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- If $\Lambda \succcurlyeq 0$ then $I_\Lambda = \ker H_\Lambda$ is a real radical ideal.
- Λ supported on points and positive iff $\Lambda = \sum_{i=1}^{r} \gamma_i \mathbf{1}_{\zeta_i}$ with $\gamma_i > 0$ and ζ_i are distinct points of $\mathbb{R}^n$.

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Let B, B' be connected to 1 of size r and $\Lambda \in \langle B^+ \cdot B'^+ \rangle^*$. The following conditions are equivalent:

1. *there exists a unique element $\tilde{\Lambda} \in R^*$ which extends Λ and such that B and B' are basis of $\mathcal{A}_\Lambda = R/I_\Lambda$.*
2. $$\mathrm{rank} H_\Lambda^{B,B'} = \mathrm{rank} H_\Lambda^{B^+,B'^+} = r.$$
3. *$H_\Lambda^{B,B'}$ is invertible and the matrices $M_i := H_\Lambda^{B,x_i B'}(H_\Lambda^{B,B'})^{-1}$ satisfy*
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☞ If $H_\Lambda^{B,B'} \succcurlyeq 0$ then $\tilde{\Lambda} \succcurlyeq 0$.

Applications

Roots with no multiplicity

Let $f = x^d + f_{d-1}x^{d-1} + \cdots + f_0 \in \mathbb{C}[x]$.

1. Compute a generic sequence $(h_i)_{0\leq i\leq 3d-3}$ such that
$$h_{d+j} = -f_{d-1}\, h_{d-j-1} - \cdots - f_0\, h_j.$$
2. Compute $(h'_i)_{0\leq i\leq 2d-2} = f' \cdot (h_j)$ such that $h'_j = \sum_{i=1}^{d} i\, h_{j+i-1}\, f_i$.
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☞ The polynomial of smallest degree of $\ker H_{f'}$ has the same roots as f, but with multiplicity 1.

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1. Compute a generic sequence $(h_i)_{0\le i\le 3d-3}$ such that
$$h_{d+j} = -f_{d-1}h_{d-j-1} - \cdots - f_0\, h_j.$$
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$$H_{f'} = (h'_{i+j})_{0\le i,j\le d-1}.$$

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☞ Fast algorithm: $\tilde{\mathcal{O}}(d)$.

Real roots of univariate polynomials

Let $f = x^d + f_{d-1}x^{d-1} + \cdots + f_0 \in \mathbb{R}[x]$.

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☞ Numerical algorithm, no good complexity bound yet;

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$$H_\Lambda := \begin{pmatrix} 1 & a & b & c \\ a & b & c & c+a-1 \\ b & c & c+a-1 & c+b-1 \\ c & c+a-1 & c+b-1 & 2\,c-1 \end{pmatrix}.$$

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- Find Λ such that $H_\Lambda \succcurlyeq 0$:

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- Compute its kernel: $\langle x-1, x^2-x, x^3-x^2, \ldots \rangle$.

Solving exponential polynomials

Given $f(u_1, \ldots, u_n) = \sum_{i=1}^{r} \lambda_i e^{u_1\gamma_{i,1}+\cdots+u_n\gamma_{i,n}} = \sum_{i=1}^{r} \lambda_i \zeta_{i,1}^{u_1} \cdots \zeta_{i,n}^{u_n}$ $(\zeta_{i,j} = e^{\gamma_{i,j}})$, find the points $(\zeta_{i,1}, \ldots, \zeta_{i,n}) \in \mathbb{C}^n$ and $\lambda_i \in \mathbb{C}$ for $i = 1 \ldots, r$.

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(Generalized Prony's method)

Example

- Let $f(u_1, u_2) = 1 + 2^{u_1+u_2} + 3^{u_2}$.
- Take $B = \{1, x_1, x_2\}$ (or $\{(0,0),(1,0),(0,1)\}$).

$$H_f^{B,B} = \begin{bmatrix} f(0,0) & f(1,0) & f(0,1) \\ f(1,0) & f(2,0) & f(1,1) \\ f(0,1) & f(1,1) & f(0,2) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 4 \\ 6 & 14 & 8 \\ 4 & 8 & 6 \end{bmatrix},$$

$$H_f^{B,x_1B} = \begin{bmatrix} 6 & 14 & 8 \\ 14 & 36 & 18 \\ 8 & 18 & 12 \end{bmatrix}.$$

- Compute the generalized eigenvectors for the eigenvalues $(1, 2, 3)$

$$V = \begin{bmatrix} 2 & -1 & 0 \\ -1/2 & 0 & 1/2 \\ -1/2 & 1 & -1/2 \end{bmatrix} \text{ and } H_f^{B,B} V = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}.$$

- This yields the roots $(1,1),(2,2),(3,1)$.

Real radical computation

Let $F = \{f_1, \ldots, f_s\} \subset \mathbb{R}[\mathbf{x}]$.
Let $S \subset R = \mathbb{R}[\mathbf{x}]$ with $1 \in S$, $G \subseteq \langle S \cdot S \rangle$, and

$$\mathcal{L}_{S,G,\succcurlyeq} := \{\Lambda \in \langle S \cdot S \rangle^* \mid \Lambda(g) = 0,\ \forall g \in G \text{ and } \Lambda(p^2) \geq 0,\ \forall p \in S\}.$$

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Repeat

1. Compute a rewriting family $G \subset (F)$ and choose S in a basis for G;
2. Compute a generic element Λ^* of $\mathcal{L}_{S,G,\succcurlyeq}$;
3. Compute $\ker H_{\Lambda^*}^{S,S}$ and add its generators to F;
4. Update the rewriting family G of F;

until $\ker H_{\Lambda^*} = \{0\}$ and G is a border basis for $B = S$.

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☞ Output a border basis of $\sqrt[\mathbb{R}]{(F)}$.

Bezoutian

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Bezoutian matrix: $B_{f_0, \ldots, f_n} = B_{f_0} = (\theta_{\alpha,\beta})_{\alpha,\beta}$.

Reduction

Apply the pencil reduction algorithm applied to $[B_1, B_{x_1}, \ldots, B_{x_n}]$ in order to obtain matrices

$$[\Delta_0, \Delta_1, \ldots, \Delta_n]$$

such that

- Δ_0 inversible,
- $\Delta_i(\mathbf{x}, \mathbf{y}) \equiv x_i \, \Delta_0(\mathbf{x}, \mathbf{y}) \mod I(\mathbf{x}) \qquad \equiv y_i \, \Delta_0(\mathbf{x}, \mathbf{y}) \mod I(\mathbf{y})$.

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☞ The linear form $\tilde{\tau}$ extending τ is the **residue** of $f_1, \ldots, f_n$.

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Thanks for your attention