Effective models for constructive mathematics

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Aim of our talk

our view - jww G. Sambin- to meet MAP goal:

$$\Downarrow$$

The objective of the MAP 2012 conference:

to bridge the gap between

conceptual (abstract) and computational (constructive) mathematics

via a computational understanding of abstract mathematics.

our view (**jww G. Sambin**)

to bridge the gap between conceptual (abstract)
and computational (constructive) mathematics
via a computational understanding of abstract mathematics.

1. develop constructive mathematics:

   take INTUITIONISTIC LOGIC + set theory

   NO CLASSIC LOGIC = NO proof by contradiction!!

2. build a foundation, actually a two-level foundation, to formalize it

CLASSICAL LOGIC =

INTUITIONISTIC LOGIC

+ DOUBLE NEGATION LAW

$$\neg\neg A \to A$$

(i.e. + proofs by contradiction)

Abstract of our talk

to meet MAP goal:

- (**jww G. Sambin**) need of a TWO LEVEL theory

  + example: our minimalist foundation

- categorical/algebraic description of the link

  between the TWO LEVELS (**jww G. Rosolini**)

- two effective/computational models for our foundation:

  - one to extract the computational contents of proofs

  - another for embedding constructive proofs in classical set theory

the need of a two-level foundation *(***jww G. Sambin***)*

from the need of putting together:

ABSTRACTION + COMPUTATIONAL IMPLEMENTATION of maths

example of abstraction: quotients!!

the need of a two-level foundation (**jww G. Sambin**)

example of levels to describe reals:

algebraic description: Archimedean complete totally ordered field
costructive description: quotient of decimal approximations of reals for ex: $1.39999999\ldots = 1.4$
computer description

what is a constructive foundation ?

ideal constructive foundation:

a *double face* theory =intuitionistic logic + set theory

+ programming language

why??: to get extraction of programs from proofs

decidable type checking for program correctness

reliable theory

$\Downarrow$

type theory provides examples

our view:

basic reliable theory $\Rightarrow$ intensional + predicative + constructive

as Martin-Löf's type theory

a predicative theory = theory with NO IMPREDICATIVE constructions

$\Longrightarrow$ for ex. power of subsets is a COLLECTION NOT a set

predicative set theory makes essential use of 2 sizes:

SETS + COLLECTIONS

why a SINGLE theory is NOT enough

ideal constructive theory: intensional + predicative + constructive

(with decidable equality of sets and elements)

\+ description abstraction/quotients

(with undecidable equality of sets and elements)

more formally: in [M.-Sambin'05] the need of two-levels follows

from consistency with MATHEMATICAL PRINCIPLES

as Axiom of Choice + Formal Church Thesis

why a SINGLE theory is NOT enough

relevant examples of constructive foundations:

Martin-Löf's intensional type theory:	- reliable programming language
	- YES explicit computational contents
	- complex setoid model to handle extensional abstractions
	- NO natural interpretation in classical ZFC theory preserving propositions

type theory: suitable for mathematicians that are logician/computer scientist

Aczel's CZF	- usual math language
(Constructive Zermelo Fraenkel set theory) :	- YES clear embedding in classical ZFC theory
	- NO explicit computational contents (needs interpretation in type theory also for its constructive reliability)

suitable for all mathematicians

first example of two-level foundation?

to meet MAP goal

Aczel's CZF (usual math language)
$\Downarrow$ (interpreted in)
Martin-Löf's type theory (reliable programming language)

use of choice principles is relevant for some axioms.

our notion of two-level foundation

from [M.-Sambin'05], [M.'09]

a constructive foundation

=

a theory with two levels

an intensional level enjoying extraction of programs from proofs

+

an extensional level obtained by ABSTRACTION from the intensional one

via a QUOTIENT completion

the link between levels	is local and modular
	preserves the logic
	follows Sambin's *forget-restore principle*

NO use of choice principles to interpret the extensional level

the two-level foundation needs an extra level!

two-level foundation	extensional level intensional level
for computer extraction	realizability level

intensional level $\neq$ realizability level

for minimality of the extensional level!

for ex: "all functions are recursive" holds at the realizability level

but canNOT be lifted at the extensional level

for compatibility with classical extensional levels

Plurality of constructive foundations $\Rightarrow$ need of a minimalist foundation

	classical	constructive
	ONE standard	NO standard
impredicative	Zermelo-Fraenkel set theory	internal theory of topoi Coquand's Calculus of Constructions
predicative	Feferman's explicit maths	Aczel's CZF Martin-Löf's type theory Feferman's constructive expl. maths

what common core ??

Aczel's CZF is not the minimal theory!

Our two level minimalist constructive foundation

from [M.-Sambin'05],[M.'09]

emTT	=	extensional minimalist level
$\Downarrow \mathcal{I}$		(interpretation via quotient completion)
mtt	=	intensional minimalist type theory
		predicative Coq

emtt $\Rightarrow$ clearly interpretable in $\begin{cases} \text{Aczel's CZF} \\ \text{Feferman's predicative classical set theory} \end{cases}$

Our two level minimalist constructive foundation

from [M.-Sambin'05],[M.'09]

emTT	=	extensional minimalist level
$\Downarrow \mathcal{I}$		(interpretation via quotient completion)
mtt	=	intensional minimalist type theory
		predicative Coq

via interpretation	$\mathcal{I}$	
extensional equality of set	=	existence of canonical isomorphisms
(undecidable)		among intensional sets (with decidable equality)

Effective models of our minimalist intensional level

$\mathrm{mtt} \longrightarrow$

(k-rea) KLEENE REALIZABILITY

Functions$(Nat, Nat) =$ all computable

INcompatible with classical predicativity

propositions as data types

for EXTRACTION of COMPUTATIONAL contents

(lo-k-rea) LOGIC ENRICHED KLEENE REALIZABILITY

Functions$(Nat, Nat) =$ NOT all computable

only Operations$(Nat, Nat) =$ all computable

for EMBEDDING in CLASSICAL predicative theory

preserving propositions

how to lift the effective models?

$$\begin{array}{ccc} \text{emtt} & \longrightarrow & ??? \\ \downarrow{\scriptstyle \mathcal{I}} & & \\ \text{mtt} & \longrightarrow & \text{k-rea} \end{array}$$

$$\begin{array}{ccc} \text{emtt} & \longrightarrow & ??? \\ \downarrow{\scriptstyle \mathcal{I}} & & \\ \text{mtt} & \longrightarrow & \text{lo-k-rea} \end{array}$$

how to lift the effective models?

by investigating the link between the levels abstractly/categorically

(**jww G. Rosolini**) with NEW notion of quotient completion

related to a doctrine

(and NOT just to a category!)

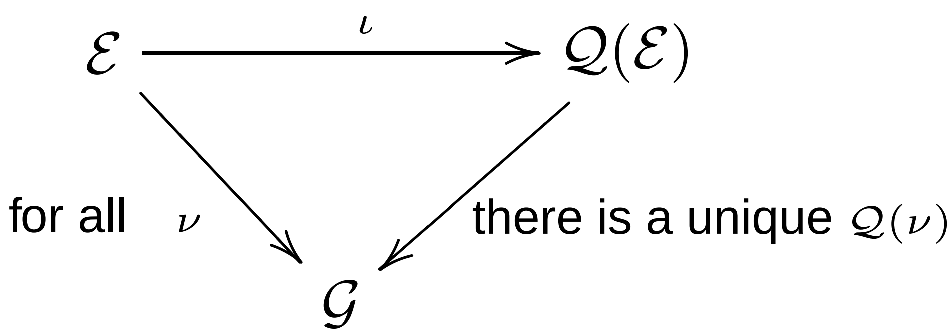
where

doctrine=	categorical interpretation of many sorted logic
	sorts are types

universal property of our quotient completion

from [M.-Rosolini'11]

Theorem: For any *elementary doctrine* $\mathcal{E}$ there is a *quotient doctrine* $Q(\mathcal{E})$ in which it embeds with $\iota : \mathcal{E} \Rightarrow Q(\mathcal{E})$ such that



uniqueness is up to natural isomorphisms

how to lift the effective models?

via the categorical quotient completion

$$\begin{array}{ccc} \text{mtt} & \longrightarrow & \text{k-rea} \\ & \Downarrow & \\ \text{emtt} & \longrightarrow & \text{Q(k-rea)} \end{array} \qquad \begin{array}{ccc} \text{mtt} & \longrightarrow & \text{lo-k-rea} \\ & \Downarrow & \\ \text{emtt} & \longrightarrow & \text{Q(lo-k-rea)} \end{array}$$

via

$$\mathcal{I} : \mathrm{emtt} \longrightarrow \mathrm{mtt}$$

that is actually

$$\text{emtt} \xrightarrow{\ \mathcal{I}\ } \text{Q(mtt)}$$

Open issues

- Describe interpretation of an extensional type theory *abstractly* in a quotient doctrine
- Extend the effective models to modelling impredicative extensions.
- Connection of our effective models with Hyland's effective topos, Joyal's arithmetic universes...

References

[M.'09] "A minimalist two-level foundation for constructive mathematic", 2009

[M.'10] "Consistency of the minimalist foundation with Church thesis and Bar Induction", 2010

[M.-Sambin'05] "Toward a minimalist foundation for constructive mathematics", 2005

[M.-Rosolini'11] "Quotient completion for the foundation of constructive mathematics", 2011