Lift-and-project hierarchies for combinatorial problems

Monique Laurent
CWI, Amsterdam & Tilburg University

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Typical combinatorial optimization problem:

$$\max \; c^T x \;\; \text{s.t. } Ax \leq b, \; x \in \{0,1\}^n$$

LP relaxation:

$$P := \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

Integral polytope to be found:

$$P_I := \mathrm{conv}(P \cap \{0,1\}^n)$$

Goal: Procedure to construct a tighter, **tractable** relaxation P' such that

$$P_I \subseteq P' \subseteq P$$

leading to P_I after **finitely many iterations.**

Cutting planes

Gomory-Chvátal closure of $P = \{x \in \mathbb{R}^n : Ax \le b\}$:

$$P' = \{x \mid u^T Ax \le \lfloor u^T b \rfloor \quad \forall u \ge 0 \text{ with } u^T A \text{ integer}\}.$$

- P' is a polyhedron.
- P_I is found after finitely many iterations. [Chvátal 1973]
- $O(n^2 \log n)$ iterations suffice if $P \subseteq [0,1]^n$. [Eisenbrand-Schulz 1999]
- **But optimization over P' is hard!** [Eisenbrand 1999]

This talk: Lift-and-project methods

We present several techniques to construct a *hierarchy* of **LP/SDP** relaxations:

$$P \supseteq P_1 \supseteq \ldots \supseteq P_n = P_I.$$

- ⇝ Balas-Ceria-Cornuéjols hierarchy [1993] — LP
- ⇝ Lovász-Schrijver N / N_+ operators [1991] — LP / SDP
- ⇝ Sherali-Adams hierarchy [1990] — LP
- ⇝ Lasserre hierarchy [2001] — SDP

Common feature:
One can **optimize in polynomial time** over P_t for any **fixed** t.

Comparison:

$$\text{SA} \subseteq \text{LS} \subseteq \text{BCC}$$

$$\text{Las} \subseteq \text{SA} \cap \text{LS}_+$$

Great interest recently in such hierarchies:

- **Polyhedral combinatorics:** How many rounds are needed to find P_I? Which valid inequalities are satisfied after t rounds? New tractable instances?

- **Proof systems:** Use hierarchies as a model to generate inequalities and show e.g. $P_I = \emptyset$.

- **Complexity theory:** What is the integrality gap after t rounds? Can one use the hierarchy to get improved tractable approximations? Link to hardness of the problem?

Common background for the hierarchies: Moment theory and sums of squares of polynomials.

Plan of the lecture

- Balas-Ceria-Cornuéjols, Lovśz-Schrijver, Sherali-Adams constructions.
- Full lifting and moment matrices
- Lasserre hierarchy
- Application to matchings, stable sets, knapsack, max-cut
- Copositive hierarchy

Some notation

$$P = \{x \in \mathbb{R}^n : Ax \le b\}$$

Homogenize P to the cone:

$$\tilde{P} = \{(x_0, x) \in \mathbb{R}^{n+1} : bx_0 - Ax \ge 0\}$$

$$= \{y \in \mathbb{R}^{n+1} : \ {g_\ell}^T y \ge 0 \ \ (\ell = 1, \cdots, m)\}$$

writing $Ax \le b$ as $a_\ell^T x \le b_\ell \quad (\ell = 1, \cdots, m)$

and setting $g_\ell = \begin{pmatrix} b_\ell \\ -a_\ell \end{pmatrix}$.

Lift-and-project strategy

1. **Generate new constraints: Multiply** the system $Ax \leq b$ by products of the constraints $x_i \geq 0$ and $1 - x_i \geq 0$.

   $\rightsquigarrow$ Polynomial system in x.

2. **Linearize** (and **lift**) by introducing new variables y_I for products $\prod_{i \in I} x_i$ and setting $x_i^2 = x_i$.

   $\rightsquigarrow$ Linear system in (x, y).

3. **Project** back on the x-variable space.

   $\rightsquigarrow$ LP relaxation P' satisfying

$$P_I \subseteq P' \subseteq P.$$

The methods vary in the choice of the multipliers and of iterating.

The Balas-Ceria-Cornuéjols construction

1. **Multiply** the system $Ax \leq b$ by x_1 and $1 - x_1$:
$$x_1(b - Ax) \geq 0, \ (1 - x_1)(b - Ax) \geq 0$$
2. **Linearize:** Set $y_i = x_1 x_i$, identify $y_1 = x_1$ and get the **lift**:
$$M_1 = \{(x, y) : \ y_1 = x_1, \ bx_1 - Ay \geq 0, \ b(1 - x_1) - A(x - y) \geq 0\}$$
3. **Project** M_1 back to the x-subspace and get P_1 such that
$$P_I \subseteq P_1 \subseteq P.$$
4. **Iterate:** use variable x_2 starting from P_1 and get P_{12}, etc.

Lemma

$P_1 \ = \ \mathrm{conv}(P \cap \{x : x_1 = 0, 1\})$.

Pf: "$\subseteq$": Write $x \in P_1$ as $x = x_1 \frac{y}{x_1} + (1 - x_1) \frac{x - y}{1 - x_1}$.
"$\supseteq$": $x \in P \cap \{x : x_1 = 0, 1\} \implies (x, x_1 x) \in M_1 \implies x \in P_1$.

Corollary

Find P_I after n steps.

The Lovász-Schrijver construction: N-operator

1. **Multiply** $Ax \le b$ by x_i, $1 - x_i \;\; \forall i \in [n]$ and get the system:

$$(b_\ell - a_\ell^T x)x_i = g_\ell^T \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T e_i \ge 0 \;\; \forall \ell,$$

$$(b_\ell - a_\ell^T x)(1 - x_i) = g_\ell^T \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T (e_0 - e_i) \ge 0 \;\; \forall \ell.$$

2. **Linearize:** The new matrix variable $Y = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T$ belongs to

$$\mathcal{M}(P) = \{Y \in \mathcal{S}_{n+1} \mid Y_{0i} = Y_{ii},\; Ye_i, Y(e_0 - e_i) \in \tilde{P} \;\; \forall i \in [n]\},$$

3. **Project:**

$$N(P) = \left\{ x \in \mathbb{R}^n \mid \exists Y \in \mathcal{M}(P) \text{ s.t. } \begin{pmatrix} 1 \\ x \end{pmatrix} = Ye_0 \right\}$$

The Lovász-Schrijver construction: N_+-operator

1. **Multiply** $Ax \leq b$ by x_i, $1 - x_i$ $\forall i \in [n]$ and get the system:

$$(b_\ell - a_\ell^T x)x_i = g_\ell^T \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T e_i \geq 0 \quad \forall \ell,$$

$$(b_\ell - a_\ell^T x)(1 - x_i) = g_\ell^T \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T (e_0 - e_i) \geq 0 \quad \forall \ell.$$

2. **Linearize:** The new matrix variable $Y = \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^T$ belongs to

$$\mathcal{M}(P) = \{ Y \in \mathcal{S}_{n+1} \mid Y_{0i} = Y_{ii},\ Ye_i, Y(e_0 - e_i) \in \tilde{P} \ \forall i \in [n] \},$$

$$\mathcal{M}_+(P) = \mathcal{M}(P) \cap \mathcal{S}_{n+1}^+.$$

3. **Project:**

$$N_+(P) = \left\{ x \in \mathbb{R}^n \mid \exists Y \in \mathcal{M}_+(P) \text{ s.t. } \begin{pmatrix} 1 \\ x \end{pmatrix} = Ye_0 \right\}$$

Properties of the N- and N_+-operators

0. Iterate: $N^t(P) = N(N^{t-1}(P))$, $N_+^t(P) = N_+(N_+^{t-1}(P))$.
1. $P_I \subseteq N_+(P) \subseteq N(P) \subseteq P$.
2. $N(P) \subseteq \bigcap_{i\in[n]} \mathrm{conv}(P \cap \{x \mid x_i = 0,1\})$.
3. $N^n(P) = P_I$.
4. If one can optimize in polynomial time over P, then the same holds for $N^t(P)$ and for $N_+^t(P)$ for any **fixed** t.

Example

For the ℓ_1-ball centered at $e/2$:

$P = \left\{x \in \mathbb{R}^V \mid \sum_{i\in I} x_i + \sum_{i\in V\setminus I}(1-x_i) \geq \frac{1}{2} \;\; \forall I \subseteq V\right\}$,

$P_I = \emptyset$, but $\frac{1}{2}e \in N_+^{n-1}(P)$.

Hence, **n iterations of the N_+ operator are needed** to find P_I.

Application to stable sets

$P = \mathrm{FR}(G) = \{x \in \mathbb{R}^V_+ \mid x_i + x_j \leq 1 \ (ij \in E)\}$
$P_I = \mathrm{STAB}(G)$: stable set polytope of $G = (V, E)$.

1. $Y \in \mathcal{M}(\mathrm{FR}(G)) \Longrightarrow y_{ij} = 0$ for all edges $ij \in E$.
2. The **clique inequality**: $\sum_{i \in Q} x_i \leq 1$ is valid for $N_+(\mathrm{FR}(G))$,

   but its N-rank is $|Q| - 2$. $\rightsquigarrow$ **SDP helps!**
3. The **odd circuit inequalities**: $\sum_{i \in V(C)} x_i \leq \frac{|C|-1}{2}$

   are valid for $N(\mathrm{FR}(G))$ and they determine it **exactly**.
4. $\frac{n}{\alpha(G)} - 2 \leq N$-rank $\leq n - \alpha(G) - 1$.
5. N_+-rank $\leq \alpha(G)$ [tight for G = line graph of K_{2p+1}]

The Sherali-Adams construction

1. **New polynomial constraints:**
   - $x^I(1-x)^{W\setminus I}(b-Ax) \geq 0$ for $I \subseteq W$ with $|W| = t$.
   - $x^I(1-x)^{U\setminus I} \geq 0$ for $I \subseteq U$ with $|U| = t+1$.

2. **Linearize & lift:** Introduce new variables y_U for all $U \in \mathcal{P}_{t+1}(V)$, setting $y_i = x_i$ $(x_i^2 = x_i)$.

3. **Project** back on x-variables space and get $\mathrm{SA}_t(P)$.

Lemma

- $\mathrm{SA}_1(P) = N(P)$.
- $\mathrm{SA}_t(P) \subseteq N^t(P)$.

Full lifting

$$x \in \{0,1\}^n \quad \rightsquigarrow \quad y^x = \left(\prod_{i \in I} x_i\right)_{I \subseteq V} \in \{0,1\}^{\mathcal{P}(V)}$$

$$y^x = (1, x_1, .., x_n, x_1 x_2, .., x_{n-1} x_n, .., \prod_{i \in V} x_i)$$

$$\rightsquigarrow \quad Y = y^x (y^x)^T = \left(\prod_{i \in I} x_i \prod_{j \in J} x_j\right)_{I,J \subseteq V}$$

If $x \in P \cap \{0,1\}^n$ then $Y = y^x (y^x)^T$ satisfies:

1. $Y(\emptyset, \emptyset) = 1.$
2. $Y(I, J)$ *depends only on* $I \cup J$ $\rightsquigarrow$ moment matrix
3. $Y \succeq 0.$
4. $g_\ell(x) Y \succeq 0$ $\rightsquigarrow$ localizing moment matrix

These conditions characterize $\mathrm{conv}(y^x : x \in P \cap \{0,1\}^n)$, thus P_I.

Full lifting via moment matrices

Definition

Given $y \in \mathbb{R}^{\mathcal{P}(V)}$ define:

1. The **moment matrix** $M_V(y) = (y_{I \cup J})_{I,J \in \mathcal{P}(V)}$.
2. The **shifted vector** $g * y = (y_I + \sum_i g_i y_{I \cup \{i\}})_{I \in \mathcal{P}(V)}$.

   [linearize $g(x)y^x = (g(x)x^I)_I$]
3. The **localizing moment matrix** $M_V(g * y)$.

Theorem

1. $\operatorname{conv}(y^x (y^x)^T : x \in P \cap \{0,1\})$ *is equal to*

   $$\Delta_P = \{y \in \mathbb{R}^{\mathcal{P}(V)} : y_\emptyset = 1,\ M_V(y) \succeq 0,\ M_V(g_\ell * y) \succeq 0 \ \ \forall \ell\}.$$

2. P_I *is the projection of* Δ_P.
3. Δ_P *is a polytope.*

Proof

Definition

Let Z be the matrix with columns y^x for $x \in \{0,1\}^n$.

Recall:

$$\Delta_P = \{y \in \mathbb{R}^{\mathcal{P}(V)} : y_\emptyset = 1,\ M_V(y) \succeq 0,\ M_V(g_\ell * y) \succeq 0\ \forall \ell\}.$$

Lemma

$$\Delta_P = \{y \in \mathbb{R}^{\mathcal{P}(V)} : y_\emptyset = 1,\ Z^{-1}y \geq 0,\ (Z^{-1}y)_J = 0 \ \ \textit{if}\ \ \chi^J \notin P\}$$

$$= \mathrm{conv}(y^x : x \in P \cap \{0,1\}^n).$$

Proof:

1. Z diagonalizes $M_V(y)$: $\quad M_V(y) = Z \ \mathrm{diag}(Z^{-1}y)\ Z^T$.

   Thus: $M_V(y) \succeq 0 \iff Z^{-1}y \geq 0$.

2. $M_V(g_\ell * y) \succeq 0 \iff (Z^{-1}y)_J\ g_\ell(\chi^J) \geq 0$ for all J.

Case $n = 2$

Z is the $0/1$ matrix indexed by $\mathcal{P}(V)$ with

$$Z(I,J) = 1, \quad Z^{-1}(I,J) = (-1)^{|J \setminus I|} \text{ if } I \subseteq J, \; 0 \text{ otherwise.}$$

$$Z = \begin{array}{c} \\ \emptyset \\ 1 \\ 2 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccc} \emptyset & 1 & 2 & 12 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \quad \rightsquigarrow \quad Z^{-1} = \begin{array}{c} \\ \emptyset \\ 1 \\ 2 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccc} \emptyset & 1 & 2 & 12 \end{array} \\ \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$M_V(y) = \begin{pmatrix} y_0 & y_1 & y_2 & y_{12} \\ y_1 & y_1 & y_{12} & y_{12} \\ y_2 & y_{12} & y_2 & y_{12} \\ y_{12} & y_{12} & y_{12} & y_{12} \end{pmatrix} \succeq 0 \iff \begin{cases} y_\emptyset - y_1 - y_2 + y_{12} \geq 0 \\ y_1 - y_{12} \geq 0 \\ y_2 - y_{12} \geq 0 \\ y_{12} \geq 0 \end{cases}$$

Example

$$M_V(y) = \begin{pmatrix} y_\emptyset & y_1 & y_2 & y_{12} \\ y_1 & y_1 & y_{12} & y_{12} \\ y_2 & y_{12} & y_2 & y_{12} \\ y_{12} & y_{12} & y_{12} & y_{12} \end{pmatrix} \succeq 0 \iff \left\{ \begin{aligned} y_\emptyset - y_1 - y_2 + y_{12} &\geq 0 \\ y_1 - y_{12} &\geq 0 \\ y_2 - y_{12} &\geq 0 \\ y_{12} &\geq 0 \end{aligned} \right.$$

Consider

$$P = \left\{ (x_1, x_2) : g(x) = \frac{3}{2} - x_1 - x_2 \geq 0 \right\}.$$

$$(g * y)_\emptyset = \frac{3}{2} y_\emptyset - y_1 - y_2, \quad (g * y)_1 = \frac{3}{2} y_1 - y_1 - y_{12} = \frac{1}{2} y_1 - y_{12},$$

$$(g * y)_2 = \frac{1}{2} y_2 - y_{12}, \quad (g * y)_{12} = \frac{3}{2} y_{12} - y_{12} - y_{12} = -\frac{1}{2} y_{12}.$$

$$(g * y)_\emptyset - (g * y)_1 - (g * y)_2 + (g * y)_{12} = \frac{3}{2}(y_\emptyset - y_1 - y_2).$$

$$M_V(y), M_V(g * y) \succeq 0 \iff y_{12} = 0, \; y_1, y_2 \geq 0, \; y_\emptyset - y_1 - y_2 \geq 0.$$

Recipe for SDP hierarchies

Get SDP hierarchies by **truncating** $M_V(y)$ and $M_V(g_\ell * y)$:

- Consider $M_U(y) = (y_{I\cup J})_{I,J\subseteq U}$, indexed by $\mathcal{P}(U)$ for $U \subseteq V$,
- or $M_t(y) = (y_{I\cup J})_{|I|,|J|\leq t}$, indexed by $\mathcal{P}_t(V)$ for some $t \leq n$.

1. **(local) Sherali-Adams relaxation** $\mathrm{SA}_t(P)$:

   $$M_U(y) \succeq 0,\ M_W(g_\ell * y) \succeq 0 \ \ \forall U \in \mathcal{P}_{t+1}(V),\ W \in \mathcal{P}_t(V).$$

   $\rightsquigarrow$ LP with variables y_I for all $I \in \mathcal{P}_{t+1}(V)$

2. **(global) Lasserre relaxation** $\mathrm{L}_t(P)$:

   $$M_t(y) \succeq 0,\ M_{t-1}(g_\ell * y) \succeq 0.$$

   $\rightsquigarrow$ SDP with variables y_I for all $I \in \mathcal{P}_{2t}(V)$

Clearly:

$$\mathrm{L}_t(P) \subseteq \mathrm{SA}_{t-1}(P).$$

Comparison

- The Lasserre hierarchy refines all other hierarchies:

$$\mathrm{L}_t(P) \subseteq N_+^{t-1}(P) \ \cap \ \mathrm{SA}_{t-1}(P).$$

- $\mathrm{L}_t(P)$ is tighter, but more expensive to compute:
  - SDP for $\mathrm{L}_t(P)$ involves one matrix of size $O(n^t)$.
  - SDP for $N_+^{t-1}(P)$ involves $O(n^{t-2})$ matrices of size $n+1$.

- The N, N_+ operators apply to P **convex**.

  SA and Lasserre apply to P **basic closed semi-algebraic**.

Application to the knapsack problem

Given $a, b, c \geq 0$:

$$\mathrm{OPT} \ = \max \ c^T x \ \text{ s.t. } \ a^T x \leq b, \ x \in \{0,1\}^n$$

$$\mathrm{LP} \ = \max \ c^T x \ \text{ s.t. } \ a^T x \leq b, \ x \in [0,1]^n.$$

$$\frac{\mathrm{LP}}{\mathrm{OPT}} \leq 2.$$

Theorem (Karlin-Mathieu-Thach Nguyen 2011)

1. *For the Sherali-Adams relaxation:* $\frac{\text{max over } SA_t}{OPT} \geq \frac{2}{1+t/n}$.
2. *For the Lasserre relaxation:* $\frac{\text{max over } L_t}{OPT} \leq 1 + \frac{1}{t-1}$.

The Lasserre hierarchy is more powerful than Sherali-Adams.

Application to the matching polytope

$G = (V, E)$.

$P = \{x \in \mathbb{R}^E_+ \mid x(\delta(v)) \leq 1 \ \forall v \in V\}$.

P_I: **matching polytope** of G, whose linear inequality description needs exponentially many inequalities.

Open question: Exist a linear or sdp lift of polynomial size?

For $G = K_{2n+1}$:

- BCC-rank $= n^2$ [Aguilera et al. 2004]
- N-rank $\in [2n, n^2]$ [LS 1991] [Goemans-Tunçel 2001]
- N_+-rank $= n$ [Stephen-Tunçel 1999]
- SA-rank $= 2n - 1$ [Mathieu-Sinclair 2009]
- Lasserre rank $\in \left[\left\lfloor \frac{n}{2} \right\rfloor, n\right]$ [Yu Hin-Tunçel 2011]

Application to stable sets

- For $t \geq 2$, $L_t(\mathrm{FR}(G))$ is obtained (by projection) from the conditions:

$$y_0 = 1,\ M_t(y) \succeq 0,\ y_{ij} = 0\ (ij \in E).$$

- $\mathrm{STAB}(G)$ is found after $t = \alpha(G)$ iterations.

- This is a natural generalization of the **theta body** $\mathrm{TH}(G)$ obtained (by projection) from the conditions:

$$y_0 = 1,\ M_1(y) \succeq 0,\ y_{ij} = 0\ (ij \in E).$$

- The **theta number** [Lovász 1979]:

$$\vartheta(G) = \max_{(y_1,\cdots,y_n)\in \mathrm{TH}(G)} \sum_{i\in V} y_i.$$

Why is $\vartheta(G)$ important?

Links structural properties of graphs & geometry of polyhedra.

$$\mathrm{QFR}(G) = \left\{x \in \mathbb{R}_+^V : \sum_{i\in Q} x_i \leq 1 \quad \forall \text{ cliques } Q \subseteq V\right\}.$$

$$\mathrm{STAB}(G) \subseteq \mathrm{TH}(G) \subseteq \mathrm{QFR}(G).$$

Theorem (Chvátal 75, Grötschel-Lovász-Schrijver 81, CRST 02)

G is **perfect***: G does not contain an induced odd circuit on at least five nodes or its complement* $\iff \mathrm{TH}(G) = \mathrm{STAB}(G)$ $\iff \mathrm{TH}(G) = \mathrm{QFR}(G)$.

For G perfect:

- $\alpha(G) = \vartheta(G)$ can be computed in **polynomial time.**
- $\mathrm{STAB}(G)$ needs exponentially many linear inequalities.
- $\mathrm{STAB}(G)$ has a psd lift of size $n+1$.
- $\mathrm{STAB}(G)$ has a linear lift of size $n^{O(\log n)}$. [Yannakakis 1991]
- **Open:** Exist linear lift of **polynomial** size?

Why is $\vartheta(G)$ useful ?

$\vartheta(G)$ gives useful bounds that **can be computed**.

- Coding theory: Maximum size of error correcting codes ?

  $\rightsquigarrow$ Wanted: $\alpha(G)$ for Hamming graphs on $\{0,1\}^n$.

  $\rightsquigarrow$ $\vartheta(G)$ is the Delsarte bound.

  $\rightsquigarrow$ Lasserre relaxation of order 2 give best known bounds.

  [Schrijver, Gijswijt, L., etc.]

- Geometric packing problems (kissing number, coloring):

  $\rightsquigarrow$ Work with infinite graphs on the Euclidean space or the unit sphere.

  [Bachoc, Vallentin, Oliveira, etc.]

On the dual side: Sums of squares representations

- The **inner (point)** description of the Lasserre relaxation $L_t(G)$:

$$y_\emptyset = 1,\ M_t(y) \succeq 0,\ y_{ij} = 0\ (ij \in E).$$

- **Outer (linear inequality)** description?

$$\textbf{ideal:}\ I = \langle x_i^2 - x_i\ (i \in V),\ x_i x_j\ (ij \in E)\rangle.$$

$$\mathrm{STAB}(G) = \mathrm{conv}(V_{\mathbb{R}}(I))$$

$$= \{x \in \mathbb{R}^n : f(x) \geq 0\ \text{ for all } \textbf{linear } f \geq 0 \text{ on } V_{\mathbb{R}}(I)\}.$$

Theorem (Gouveia-Parrilo-Thomas 2011)

1. $L_t(G) = \{x \in \mathbb{R}^n : f(x) \geq 0$ *for all* **linear** $f \in \Sigma_{2t} + I\}$.
2. *G is perfect $\Longleftrightarrow$ Any linear $f \geq 0$ on $V_{\mathbb{R}}(I)$ belongs to $\Sigma_2 + I$.*

Application to Max-Cut

Max-Cut: $\max \sum_{ij \in E} w_{ij}(1 - x_i x_j)/2$ s.t. $x \in \{\pm 1\}^n$.

Cut polytope: $\mathrm{CUT}_n = \mathrm{conv}(xx^T : x \in \{\pm 1\}^n)$.

- The Lasserre relaxation of order 1:

$$L_1 = \{X \in \mathcal{S}^n : X \succeq 0,\ X_{ii} = 1\ (i \in V)\}.$$

- This is the SDP used by [Goemans-Williamson 1995] for their celebrated **0.878-approximation algorithm.**
- This is the first (and only) improvement on the easy 0.5-approximation algorithm.
- Best possible under the *unique games conjecture* (if P$\neq$NP).

Higher order relaxations

- L_t is defined by the conditions:

$$y_\emptyset = 1, \; M_t(y) = (y_{I\Delta J})_{I,J\in\mathcal{P}_t(V)} \succeq 0.$$

- L_2 satisfies the triangle inequalities: $x_{ij} + x_{ik} + x_{jk} \geq -1.$
- L_{t+1} satisfies the $(2t+1)$-point inequalities: [La 2001]

$$\sum_{1\leq i<j\leq 2t+1} x_{ij} \geq -t.$$

But L_t **does not**. [La 2003]

- **Hence:** the Lasserre rank of $\mathrm{CUT}(K_n)$ is at least $\lceil n/2 \rceil$.

Open: Does equality hold? [**Yes** for $n \leq 7$]

Theorem (Fiorini-Massar-Pokutta-de Wolf 2011)

The smallest size of a linear lift of CUT_n *is* $2^{\Omega(n)}$.

Open: What about PSD lifts?

Another hierarchy: via copositive programming

Theorem (de Klerk-Pasechnik 2002)

$\alpha(G) = \min\ \lambda\ \ s.t.\ \lambda(I + A_G) - J \in \mathcal{C}_n$.

Definition

$\mathcal{C}_n$: cone of **copositive** matrices M, i.e., $x^T Mx \geq 0$ for all $x \geq 0$.

Idea [Parrilo 2000]: Replace $\mathcal{C}_n$ by the subcones:

$$\mathcal{K}_n^{(t)} = \left\{ M \in \mathcal{S}_n \mid \Big(\sum_{i,j=1}^{n} M_{ij} x_i^2 x_j^2 \Big) \Big(\sum_{i=1}^{n} x_i^2 \Big)^t \text{ is SOS} \right\},$$

Theorem (Pólya)

If M is strictly copositive, then $(x^T Mx)(\sum_{i=1}^n x_i)^r$ has non-negative coefficients, and thus $M \in \bigcup_{t\geq 0} \mathcal{K}_n^{(t)}$.

SDP bound: $\vartheta^{(t)}(G) = \min\ \lambda\ \text{ s.t. } \lambda(I + A_G) - J \in \mathcal{K}_n^{(t)}$.

- The Lasserre hierarchy refines the copositive hierarchy:

$$\text{max over } L_{t+1}(G)\ \leq\ \vartheta^{(t)}(G).$$

- The Lasserre hierarchy converges in $\alpha(G)$ steps.

Conjecture (de Klerk-Pasechnik 2002)

The copositive hierarchy converges in $\alpha(G) - 1$ steps:

$$\left(\alpha(G)\left(\sum_i x_i^4 + 2\sum_{ij\in E} x_i^2 x_j^2\right) - (\sum_i x_i^2)^2\right)\left(\sum_i x_i^2\right)^{\alpha(G)-1} \in \Sigma.$$

Theorem (Gvozdenovic-La 2007)

Yes: *For graphs with $\alpha(G) \leq 8$.*